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Quantum thetas on noncommutative $\mathbb{T}^4$ from embeddings into lattice

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Abstract
In this paper, we investigate the theta vector and quantum theta function over noncommutative $\mathbb{T}^4$ from the embedding of $\mathbb{R} \times \mathbb{Z}^2$. Manin has constructed the quantum theta functions from the lattice embedding into vector space ($\times$ finite group). We extend Manin's construction of the quantum theta function to the embedding of vector space $\times$ lattice case. We find that the holomorphic theta vector exists only over the vector space part of the embedding, and over the lattice part we can only impose the condition for the Schwartz function. The quantum theta function built on this partial theta vector satisfies the requirement of the quantum theta function. However, two subsequent quantum translations from the embedding into the lattice part are nonadditive, contrary to the additivity of those from the vector space part.

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1. Introduction

In the quantization of a classical theta function, we encounter two types of objects. One is the theta vector introduced by Schwarz [1], which is a holomorphic element of a projective module over a unitary quantum torus. The other is the quantum theta function introduced by Manin [2–5], which is an element of the function ring of the quantum torus itself. This is a natural outcome if one considers the process of quantization, in which commutative physical observables become operators acting on the states. Classically one deals with one type of objects, observables. After quantization one deals with two types of objects, operators and states. That is what happens here. In the classical sense, a set of specific values of observables constitutes a state. The (classical) theta function is just like a state function. On the other hand, the quantum theta function and theta vector correspond to an operator and a state vector, respectively, in the quantum sense. Manin [4, 5] has defined the quantum theta function via the Rieffel's algebra-valued inner product [6] of a theta vector [7] from the embedding of the type $\mathbb{R}^p(\times F)$ for the quantum torus. Here, $d = 2p$ is the dimension of the relevant quantum torus and F is a finite group. In [6], it was shown that the general embedding for the quantum torus is of the type $\mathbb{R}^p \times \mathbb{Z}^q(\times F)$, where $d = 2p+q$ is the dimension of the relevant quantum torus. Manin has constructed the quantum theta functions only for the embeddings of $\mathbb{R}^p$ type, and those from the $\mathbb{R}^p \times \mathbb{Z}^q$ type have been left in question [5].

One needs to know the result of the $\mathbb{Z}^q$ type embedding in order to understand the full symmetry of quantum tori including the Morita equivalence. In [8], the symmetry of the quantum torus was investigated, restricted to the symmetry of the algebra and its module, not related to the Morita equivalence. In this paper, we construct the quantum theta functions in a more general $\mathbb{R}^p \times \mathbb{Z}^q$ type of embedding that Manin did not investigate. We first investigate the existence of the theta vector in this setup, and find that the holomorphic theta vector does not exist in the exact sense. It turns out that one can only construct partially holomorphic theta vectors, which are holomorphic for the embedding into the vector space ($\mathbb{R}^p$) part but not for the lattice ($\mathbb{Z}^q$) part. We then investigate whether the quantum theta function satisfying the Manin's requirement can be constructed with this partially holomorphic theta vector. We find that the answer is yes.

The organization of this paper is as follows. In section 2, we construct a module for the quantum 4-torus with the embedding of $\mathbb{R}^p \times \mathbb{Z}^q$ type. In section 3, we construct the quantum theta function evaluating the scalar product of the above module, and check the Manin's requirement for the quantum theta function. In section 4, we conclude with the discussion.

2. Lattice embedding of the quantum torus

Here, we first review the embedding of the quantum torus [6] and an explicit construction of the module with an embedding of the type $\mathbb{R}^p(\times F)$ which was done for the 4-torus case in [9]. Then we construct the module with an embedding of the type $\mathbb{R}^p \times \mathbb{Z}^q(\times F)$ for the quantum 4-torus.

Quantum torus $\mathbb{T}^d_\theta$ is a deformed algebra of the algebra of smooth functions on the torus $\mathbb{T}^d$ with the deformation parameter θ, which is a real $d \times d$ antisymmetric matrix. This algebra is generated by operators $U_1, \ldots, U_d$ obeying the following relations:

$$U_j U_i = \mathrm{e}^{2\pi \mathrm{i}\theta_{ij}} U_i U_j \qquad \text{and} \qquad U_i^* U_i = U_i U_i^* = 1, \qquad i, j = 1, \ldots, d.$$

The above relations define the representation of the involutive algebra

$$\mathcal{A}^d_\theta = \left\{ \sum a_{i_1\cdots i_d} U_1^{i_1} \cdots U_d^{i_d} \,\middle|\, a = (a_{i_1\cdots i_d}) \in \mathcal{S}(\mathbb{Z}^d) \right\},$$

where $\mathcal{S}(\mathbb{Z}^d)$ is the Schwartz space of sequences with rapid decay.

Every projective module over a smooth algebra $\mathcal{A}^d_\theta$ can be represented by a direct sum of modules of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$, the linear space of Schwartz functions on $\mathbb{R}^p \times \mathbb{Z}^q \times F$, where $2p+q = d$ and F is a finite Abelian group. The module action is specified by operators on $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ and the commutation relation of these operators should be matched with that of elements in $\mathcal{A}^d_\theta$.

Recall that there is the dual action of the torus group $\mathbb{T}^d$ on $\mathcal{A}^d_\theta$ which gives a Lie group homomorphism of $\mathbb{T}^d$ into the group of automorphisms of $\mathcal{A}^d_\theta$. Its infinitesimal form generates a homomorphism of Lie algebra L of $\mathbb{T}^d$ into Lie algebra of derivations of $\mathcal{A}^d_\theta$. Note that the Lie algebra L is Abelian and is isomorphic to $\mathbb{R}^d$. Let $\delta : L \to \mathrm{Der}\big(\mathcal{A}^d_\theta\big)$ be the homomorphism. For each $X \in L$, $\delta(X) := \delta_X$ is a derivation i.e., for $u, v \in \mathcal{A}^d_\theta$,

$$\delta_X(uv) = \delta_X(u)v + u\delta_X(v). \qquad (1)$$

Derivations corresponding to the generators $\{e_1, \ldots, e_d\}$ of L will be denoted by $\delta_1, \ldots, \delta_d$. For the generators U_i's of $\mathbb{T}^d_\theta$, it has the following property:

$$\delta_i(U_j) = 2\pi \mathrm{i}\delta_{ij} U_j. \tag{2}$$

Let D be a lattice in $\mathcal{G} = \mathcal{M} \times \widehat{\mathcal{M}}$, where $M = \mathbb{R}^p \times \mathbb{Z}^q \times F$ and $\widehat{M}$ is its dual. Let Φ be an embedding map such that D is the image of $\mathbb{Z}^d$ under the map Φ. This determines a projective module to be denoted by E [6]. If E is a projective $\mathcal{A}^d_\theta$-module, a connection ∇ on E is a linear map from E to $E \otimes L^*$ such that for all $X \in L$,

$$\nabla_X(\xi u) = (\nabla_X \xi)u + \xi \delta_X(u), \qquad \xi \in E, u \in \mathcal{A}^d_\theta. \tag{3}$$

It is easy to see that

$$[\nabla_i, U_j] = 2\pi \mathrm{i}\delta_{ij} U_j. \tag{4}$$

In the Heisenberg representation the operators are defined by

$$\mathcal{U}_{(m,\hat{s})} f(r) = \mathrm{e}^{2\pi \mathrm{i}\langle r,\hat{s}\rangle} f(r+m) \tag{5}$$

for $(m, \hat{s}) \in D$, $r \in M$.

2.1. Embedding into vector space

We now review the explicit construction of a module over noncommutative $\mathbb{T}^4$ with the embedding of the type $\mathbb{R}^2(\times F)$ [9].

For the real part, we choose our embedding map as

$$\Phi_{\mathrm{inf}} = \begin{pmatrix} \theta_1 + \frac{n_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & \theta_2 + \frac{n_2}{m_2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv (x_{ij}). \tag{6}$$

Then using the previous expression for the Heisenberg representation,

$$(V_i f)(s_1, s_2) = (V_{e_i} f)(s_1, s_2) := \exp(2\pi \mathrm{i}(s_1 x_{3i} + s_2 x_{4i})) f(s_1 + x_{1i}, s_2 + x_{2i}), \qquad s_1, s_2 \in \mathbb{R},$$

we get

$$\begin{aligned} (V_1 f)(s_1, s_2) &= f\left(s_1 + \theta_1 + \frac{n_1}{m_1}, s_2\right), \\ (V_2 f)(s_1, s_2) &= \exp(2\pi \mathrm{i} s_1) f(s_1, s_2), \\ (V_3 f)(s_1, s_2) &= f\left(s_1, s_2 + \theta_2 + \frac{n_2}{m_2}\right), \\ (V_4 f)(s_1, s_2) &= \exp(2\pi \mathrm{i} s_2) f(s_1, s_2). \end{aligned}$$

For the finite part, let $F = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$, where $\mathbb{Z}_{m_i} = \mathbb{Z}/m_i\mathbb{Z}$ $(i = 1, 2)$ and consider the space $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2}$ as the space of functions on $C(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$. For all $m_i \in \mathbb{Z}$ and $n_i \in \mathbb{Z}/m_i\mathbb{Z}$ such that m_i and n_i are relatively prime, we define the operators W_i on $C(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$ corresponding to our embedding map

$$\Phi_{\mathrm{fin}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{n_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{n_2}{m_2} \end{pmatrix} \tag{7}$$

with $k_i \in \mathbb{Z}_{m_i}$ $(i = 1, 2)$ as follows:

$$\begin{aligned}
(W_1 f)(k_1, k_2) &= f(k_1 - 1, k_2),\\
(W_2 f)(k_1, k_2) &= \exp\left(2\pi \mathrm{i}\frac{n_1 k_1}{m_1}\right) f(k_1, k_2),\\
(W_3 f)(k_1, k_2) &= f(k_1, k_2 - 1),\\
(W_4 f)(k_1, k_2) &= \exp\left(2\pi \mathrm{i}\frac{n_2 k_2}{m_2}\right) f(k_1, k_2).
\end{aligned}$$

Now, we define operators $U_i = V_i \otimes W_i$ acting on the space $E := \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2}$ as

$$\begin{aligned}
(U_1 f)(s_1, s_2, k_1, k_2) &= f\left(s_1 + \theta_1 + \frac{n_1}{m_1}, s_2, k_1 - 1, k_2\right),\\
(U_2 f)(s_1, s_2, k_1, k_2) &= \mathrm{e}^{2\pi \mathrm{i}(s_1 + \frac{n_1 k_1}{m_1})} f(s_1, s_2, k_1, k_2),\\
(U_3 f)(s_1, s_2, k_1, k_2) &= f\left(s_1, s_2 + \theta_2 + \frac{n_2}{m_2}, k_1, k_2 - 1\right),\\
(U_4 f)(s_1, s_2, k_1, k_2) &= \mathrm{e}^{2\pi \mathrm{i}(s_2 + \frac{n_2 k_2}{m_2})} f(s_1, s_2, k_1, k_2).
\end{aligned} \tag{8}$$

One can see that they satisfy

$$U_2 U_1 = \mathrm{e}^{2\pi \mathrm{i}\theta_1} U_1 U_2 \qquad U_4 U_3 = \mathrm{e}^{2\pi \mathrm{i}\theta_2} U_3 U_4 \tag{9}$$

and otherwise $U_i U_j = U_j U_i$.

2.2. Embedding into the lattice

Here, we do a similar construction for the embedding of the type $\mathbb{R}^p \times \mathbb{Z}^q (\times F)$. The embedding of the finite part can be done in the exactly same manner as in the previous subsection. Thus we will suppress the expression for the finite part for brevity, and only consider the infinite part with the embedding of the type $\mathbb{R}^p \times \mathbb{Z}^q$ with $p = 1$ and $q = 2$. Here, we embed $D \subset \mathbb{R}^4$ into $\mathbb{R} \times \mathbb{Z}^2 \times \mathbb{R}^* \times \mathbb{T}^2$, and we choose our embedding as follows:

$$\Phi_{\mathrm{inf}} = \begin{pmatrix} \theta_1 & 0 & 0 & 0\\ 0 & 0 & m_{11} & m_{12}\\ 0 & 0 & m_{21} & m_{22}\\ 0 & 1 & 0 & 0\\ 0 & 0 & \hat{\delta}_{11} & \hat{\delta}_{12}\\ 0 & 0 & \hat{\delta}_{21} & \hat{\delta}_{22} \end{pmatrix} \equiv (x_{ij}), \tag{10}$$

where $\theta_1 \in \mathbb{R}$, and $m_{nl} \in \mathbb{Z}$, $\hat{\delta}_{nl} \in \mathbb{T}$ for $n, l = 1, 2$, and $i = 1, \ldots, 6$, $j = 1, \ldots, 4$. Then, the operators U_j acting on the space $E := \mathcal{S}(\mathbb{R} \otimes \mathbb{Z}^2)$ can be defined as

$$\begin{aligned}
(U_1 f)(s, n_1, n_2) &= f(s + \theta_1, n_1, n_2),\\
(U_2 f)(s, n_1, n_2) &= \mathrm{e}^{2\pi \mathrm{i}s} f(s, n_1, n_2),\\
(U_3 f)(s, n_1, n_2) &= \mathrm{e}^{2\pi \mathrm{i}(\hat{\delta}_{11} n_1 + \hat{\delta}_{21} n_2) + \pi \mathrm{i}(m_{11}\hat{\delta}_{11} + m_{21}\hat{\delta}_{21})} f(s, n_1 + m_{11}, n_2 + m_{21}),\\
(U_4 f)(s, n_1, n_2) &= \mathrm{e}^{2\pi \mathrm{i}(\hat{\delta}_{12} n_1 + \hat{\delta}_{22} n_2) + \pi \mathrm{i}(m_{12}\hat{\delta}_{12} + m_{22}\hat{\delta}_{22})} f(s, n_1 + m_{12}, n_2 + m_{22}),
\end{aligned} \tag{11}$$

where $s \in \mathbb{R}$, $n_l \in \mathbb{Z}$ for $l = 1, 2$. In the above definition of U_i operators, an extra phase term is added to conform with Manin's definition of the quantum theta function [5]. The above can be compactly written as

$$(U_j f)(s, n_1, n_2) = \mathrm{e}^{2\pi \mathrm{i}(s x_{4j} + n_1 x_{5j} + n_2 x_{6j}) + \pi \mathrm{i}(\sum_{k=1}^{3} x_{kj} x_{(k+3)j})} f(s + x_{1j}, n_1 + x_{2j}, n_2 + x_{3j}) \tag{12}$$

for $j = 1, \dots, 4$.

The commutation relations among U_i's are given by

$$U_2 U_1 = \mathrm{e}^{2\pi \mathrm{i}\theta_{12}} U_1 U_2, \qquad U_4 U_3 = \mathrm{e}^{2\pi \mathrm{i}\theta_{34}} U_3 U_4, \tag{13}$$

where $\theta_{12} = \theta_1$, $\theta_{34} = m_{11}\hat{\delta}_{12} + m_{21}\hat{\delta}_{22} - m_{12}\hat{\delta}_{11} - m_{22}\hat{\delta}_{21}$ and otherwise $U_i U_j = U_j U_i$.

3. Quantum thetas

In this section, we first define connections with complex structures for the two embedding cases in the previous section and consider the theta vector in each case. Then, we define the quantum theta function for each case following the Manin's construction.

3.1. Theta vectors

In the previous section, connections on a projective $\mathcal{A}_\theta^d$-module satisfy condition (4) and it can be written as

$$U_j \nabla_i U_j^{-1} = \nabla_i - 2\pi \mathrm{i}\delta_{ij}. \tag{14}$$

With this condition in mind, now we construct the theta vector for each embedding case.

3.1.1. Embedding into vector space. For the embedding of the type $\mathbb{R}^2(\times F)$, the above relation is satisfied, if we set

$$(\nabla_i f)(s_1, s_2) = -2\pi \mathrm{i} A_{i1} s_1 f(s_1, s_2) - 2\pi \mathrm{i} A_{i2} s_2 f(s_1, s_2) + A_{i3} \frac{\partial f(s_1, s_2)}{\partial s_1} + A_{i4} \frac{\partial f(s_1, s_2)}{\partial s_2},$$

where $A_{ik} \in \mathbb{R}$ are constants to be determined. If we denote the embedding map as $\Phi_{\mathrm{inf}} \equiv (x_{ij})$ and suppress the finite part, then U_i action can be compactly expressed as

$$(U_i f)(s_1, s_2) = \mathrm{e}^{2\pi \mathrm{i}(s_1 x_{3i} + s_2 x_{4i})} f(s_1 + x_{1i}, s_2 + x_{2i}). \tag{15}$$

Condition (14) is satisfied if

$$x_{1i} x_{3i} + x_{2i} x_{4i} = 0 \tag{16}$$

and

$$A_{ik} = \left(\Phi_{\mathrm{inf}}^{-1}\right)_{ik}. \tag{17}$$

Note that the above U_i action (15) would have an extra phase term in the Manin's convention. However, the extra term, $x_{1i}x_{3i} + x_{2i}x_{4i}$, has no contribution here due to condition (16). Incorporating the effect of the finite part, we slightly change the expression for the embedding map for the infinite part (6) as follows:

$$\Phi_{\mathrm{inf}} = \begin{pmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & 0 & \theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv (x_{ij}). \tag{18}$$

Then, conditions (16) and (17) give

$$(A_{ik}) = \begin{pmatrix} \frac{1}{\theta_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\theta_2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the following operators specify a constant curvature connection of right $\mathbb{T}^4_\theta$-module:

$$\begin{aligned} \nabla_1 &= -\frac{2\pi \mathrm{i} s_1}{\theta_1}, \qquad & \nabla_2 &= \frac{\partial}{\partial s_1}, \\ \nabla_3 &= -\frac{2\pi \mathrm{i} s_2}{\theta_2}, \qquad & \nabla_4 &= \frac{\partial}{\partial s_2}. \end{aligned} \tag{19}$$

The complexified connection space can be decomposed as a sum of a holomorphic part and an antiholomorphic part. A complex structure on the module E can be introduced by choosing the antiholomorphic subspace spanned by the following connection:

$$\overline{\nabla}_1 = \lambda_{11}\nabla_1 + \lambda_{12}\nabla_2 + \lambda_{13}\nabla_3 + \lambda_{14}\nabla_4, \qquad \overline{\nabla}_2 = \lambda_{21}\nabla_1 + \lambda_{22}\nabla_2 + \lambda_{23}\nabla_3 + \lambda_{24}\nabla_4,$$

where $\lambda_{ij} \in \mathbb{C}$. Choosing an appropriate basis such that (λ_{ij}) becomes

$$\begin{pmatrix} \lambda_{12} & \lambda_{14} \\ \lambda_{22} & \lambda_{24} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \end{pmatrix} = \begin{pmatrix} \tau_{11} & 1 & \tau_{12} & 0 \\ \tau_{21} & 0 & \tau_{22} & 1 \end{pmatrix}$$

the (2×2) matrix $\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$, $\tau_{ij} \in \mathbb{C}$, represents the complex structure of $\mathbb{T}^4_\theta$-module.

Now we consider holomorphic vectors in $\mathbb{T}^4_\theta$-module. A vector $f \in E$ is called holomorphic [1] if it satisfies

$$\overline{\nabla}_i f = 0 \qquad \text{for} \quad i = 1, 2. \tag{20}$$

The above holomorphic condition for $f \in E$ now takes the form

$$\left(\frac{2\pi \mathrm{i}\tau_{11}}{\theta_1} s_1 + \frac{2\pi \mathrm{i}\tau_{12}}{\theta_2} s_2\right) f = \frac{\partial f}{\partial s_1}, \qquad \left(\frac{2\pi \mathrm{i}\tau_{21}}{\theta_1} s_1 + \frac{2\pi \mathrm{i}\tau_{22}}{\theta_2} s_2\right) f = \frac{\partial f}{\partial s_2}. \tag{21}$$

In order for the two equations in (21) to be consistent τ_{ij} should satisfy

$$\frac{\tau_{12}}{\theta_2} = \frac{\tau_{21}}{\theta_1}. \tag{22}$$

If $\mathrm{Im}\,\Omega > 0$, equation (21) has a solution, the so-called theta vector [1, 7] on noncommutative $\mathbb{T}^4$,

$$f(s_1, s_2) = \exp[\pi \mathrm{i} S^t \Omega S], \tag{23}$$

where $S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $s_i \in \mathbb{R}$, $i = 1, 2$ and

$$\Omega = \begin{pmatrix} \frac{\tau_{11}}{\theta_1} & \frac{\tau_{12}}{\theta_2} \\ \frac{\tau_{21}}{\theta_1} & \frac{\tau_{22}}{\theta_2} \end{pmatrix}.$$

3.1.2. Embedding into a lattice. For the embedding of the type $\mathbb{R} \times \mathbb{Z}^2$ the relation (14) is satisfied if we let

$$\begin{aligned}(\nabla_i f)(s, n_1, n_2) = &-2\pi \mathrm{i} B_{i1} s f(s, n_1, n_2) - 2\pi \mathrm{i} B_{i2} n_1 f(s, n_1, n_2) \\ &- 2\pi \mathrm{i} B_{i3} n_2 f(s, n_1, n_2) + B_{i4} \frac{\partial f(s, n_1, n_2)}{\partial s}, \qquad \text{for} \quad i = 1, \ldots, 4, \end{aligned} \tag{24}$$

where $B_{ik} \in \mathbb{R}$ are constants satisfying the following condition:

$$B_{i1} x_{1j} + B_{i2} x_{2j} + B_{i3} x_{3j} + B_{i4} x_{4j} = \delta_{ij}, \qquad i, j = 1, \ldots, 4, \tag{25}$$

while x_{ij}'s in (10) should satisfy the following condition:

$$x_{1j} x_{4j} + x_{2j} x_{5j} + x_{3j} x_{6j} = 0, \qquad j = 1, \ldots, 4. \tag{26}$$

The embedding map (10) satisfies condition (26), and condition (25) gives

$$(B_{ik}) = \begin{pmatrix} \frac{1}{\theta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^{-1}. \tag{27}$$

Therefore the following operators specify a constant curvature connection of right $\mathbb{T}^4_\theta$-module E:

$$\begin{aligned} &\nabla_1 = -\frac{2\pi \mathrm{i} s}{\theta_1}, \qquad \nabla_2 = \frac{\partial}{\partial s}, \\ &\nabla_3 = -2\pi \mathrm{i}(b_{11} n_1 + b_{12} n_2), \qquad \nabla_4 = -2\pi \mathrm{i}(b_{21} n_1 + b_{22} n_2). \end{aligned} \tag{28}$$

A complex structure on the module E might be introduced in the same manner as in the previous case:

$$\overline{\nabla}_1 = \lambda_{11} \nabla_1 + \lambda_{12} \nabla_2 + \lambda_{13} \nabla_3 + \lambda_{14} \nabla_4, \qquad \overline{\nabla}_2 = \lambda_{21} \nabla_1 + \lambda_{22} \nabla_2 + \lambda_{23} \nabla_3 + \lambda_{24} \nabla_4,$$

where $\lambda_{ij} \in \mathbb{C}$. And choosing an appropriate basis (λ_{ij}) can be expressed as

$$\begin{pmatrix} \lambda_{13} & \lambda_{14} \\ \lambda_{23} & \lambda_{24} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{12} & 1 & 0 \\ \tau_{21} & \tau_{22} & 0 & 1 \end{pmatrix},$$

with the (2×2) matrix $\left(\begin{smallmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{smallmatrix}\right)$, $\tau_{ij} \in \mathbb{C}$ representing the complex structure of $\mathbb{T}^4_\theta$-module.

To be a holomorphic vector in $\mathbb{T}^4_\theta$-module, $f \in E$ now takes the form

$$\begin{aligned} &\left(\frac{2\pi \mathrm{i} \tau_{11}}{\theta_1} s + 2\pi \mathrm{i}(b_{11} n_1 + b_{12} n_2) \right) f = \tau_{12} \frac{\partial f}{\partial s}, \\ &\left(\frac{2\pi \mathrm{i} \tau_{21}}{\theta_1} s + 2\pi \mathrm{i}(b_{21} n_1 + b_{22} n_2) \right) f = \tau_{22} \frac{\partial f}{\partial s}. \end{aligned} \tag{29}$$

In order for the two equations in (29) to be consistent, τ_{ij}, b_{ij} should satisfy

$$\frac{\tau_{11}}{\tau_{12}} = \frac{\tau_{21}}{\tau_{22}}, \qquad b_{12} = \frac{\tau_{12}}{\tau_{22}} b_{22}, \qquad b_{21} = \frac{\tau_{22}}{\tau_{12}} b_{11}.$$

However, the above result yields

$$\det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = 0,$$

which is contradictory to the assumption that (b_{ij}) is the inverse matrix of (m_{ij}), the relation (27).

The above shows that one cannot have a holomorphic vector over totally complexified $\mathbb{T}^4_\theta$ in the embedding of $\mathbb{R} \times \mathbb{Z}^2$. This can be remedied by giving a complex structure only over the continuous part of the embedding space, i.e., by giving a complex structure to the connection components over $\mathbb{R} \times \mathbb{R}^*$. Now, we implement this as follows:

$$\overline{\nabla}_1 = \tau\nabla_1 + \nabla_2, \qquad \overline{\nabla}_2 = \nabla_3, \qquad \overline{\nabla}_3 = \nabla_4, \tag{30}$$

where $\tau \in \mathbb{C}$ is a complex structure constant over $\mathbb{R} \times \mathbb{R}^*$. Then, the holomorphic vectors over this complex structure satisfy

$$\overline{\nabla}_1 f(s, n_1, n_2) = 0, \tag{31}$$

which is

$$-\frac{2\pi \mathrm{i}\tau}{\theta_1} s f + \frac{\partial f}{\partial s} = 0.$$

Since f belongs to $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{Z}^2)$, $f(s, n_1, n_2)$ satisfying (31) can be given by

$$f(s, n_1, n_2) = \exp\left(\frac{\pi \mathrm{i}\tau}{\theta_1} s^2\right) g(n_1, n_2), \tag{32}$$

where $g(n_1, n_2) \in \mathcal{S}(\mathbb{Z}^2)$ is a Schwartz function. For the function $g(n_1, n_2)$, we will use a simple Schwartz function such that $f(s, n_1, n_2)$ can be expressed as

$$f(s, n_1, n_2) = \exp\left[\pi \mathrm{i}\frac{\tau}{\theta_1} s^2 - \pi\frac{1}{\theta_2}(n_1^2 + n_2^2)\right], \tag{33}$$

where $\operatorname{Im}\tau > 0$, and $\theta_1 = \theta_{12} > 0$, $\theta_2 = \theta_{34} > 0$ are given in (13).

3.2. Quantum theta functions

Before considering the quantum theta function, we first review the algebra-valued inner product on a bimodule after Rieffel [6]. Let M be any locally compact Abelian group, and $\widehat{M}$ be its dual group, and let $\mathcal{G} \equiv M \times \widehat{M}$. Let π be a representation of $\mathcal{G}$ on $L^2(M)$ such that

$$\pi_x \pi_y = \alpha(x, y)\pi_{x+y} = \alpha(x, y)\overline{\alpha}(y, x)\pi_y\pi_x \qquad \text{for} \quad x, y \in \mathcal{G}, \tag{34}$$

where α is a map $\alpha : \mathcal{G} \times \mathcal{G} \to \mathbb{C}^*$ satisfying

$$\alpha(x, y) = \alpha(y, x)^{-1}, \qquad \alpha(x_1 + x_2, y) = \alpha(x_1, y)\alpha(x_2, y),$$

and $\overline{\alpha}$ denotes the complex conjugation of α. Let D be a discrete subgroup of $\mathcal{G}$. We define $\mathcal{S}(\mathcal{D})$ as the space of Schwartz functions on D. For $\Psi \in \mathcal{S}(\mathcal{D})$, it can be expressed as $\Psi = \sum_{w\in D} \Psi(w)e_{D,\alpha}(w)$ where $e_{D,\alpha}(w)$ is a delta function with support at w and obeys the following relation:

$$e_{D,\alpha}(w_1)e_{D,\alpha}(w_2) = \alpha(w_1, w_2)e_{D,\alpha}(w_1 + w_2). \tag{35}$$

For the Schwartz functions $f, g \in \mathcal{S}(\mathcal{M})$, the algebra $(\mathcal{S}(\mathcal{D}))$-valued inner product is defined as

$${}_D\langle f, g\rangle \equiv \sum_{w\in D} {}_D\langle f, g\rangle(w)e_{D,\alpha}(w), \tag{36}$$

where

$$_D\langle f, g\rangle(w) = \langle f, \pi_w g\rangle.$$

Here, the scalar product of the type $\langle f, p\rangle$ above with $f, p \in L^2(M)$ denotes the following:

$$\langle f, p\rangle = \int f(x_1)\overline{p(x_1)}\,\mathrm{d}\mu_{x_1} \qquad \text{for} \qquad x = (x_1, x_2) \in M \times \widehat{M}, \tag{37}$$

where μ_{x_1} represents the Haar measure on M and $\overline{p(x_1)}$ denotes the complex conjugation of $p(x_1)$. The $\mathcal{S}(\mathcal{D})$-valued inner product can be represented as

$$_D\langle f, g\rangle = \sum_{w\in D}\langle f, \pi_w g\rangle e_{D,\alpha}(w). \tag{38}$$

For $\Psi \in \mathcal{S}(\mathcal{D})$ and $f \in \mathcal{S}(\mathcal{M})$, then $\pi(\Psi)f \in \mathcal{S}(\mathcal{M})$ can be written as [6]

$$(\pi(\Psi)f)(m) = \sum_{w\in D}\Psi(w)(\pi_w f)(m), \tag{39}$$

where $m \in M$, $w \in D \subset M \times \widehat{M}$.

3.2.1. Embedding into vector space. Now, we consider Manin's quantum theta function Θ_D [3–5] for the embedding into vector space. In [5], the quantum theta function was defined via algebra-valued inner product up to a constant factor [11],

$$_D\langle f, f\rangle \sim \Theta_D, \tag{40}$$

where f used in Manin's construction [5] was a simple Gaussian theta vector

$$f = \mathrm{e}^{\pi \mathrm{i} x_1^t T x_1}, \qquad x_1 \in M. \tag{41}$$

Here T is a complex structure given by a complex skew symmetric matrix. With a given complex structure T, a complex variable $\underline{x} \in \mathbb{C}^n$ can be introduced via

$$\underline{x} \equiv T x_1 + x_2, \tag{42}$$

where $x = (x_1, x_2) \in M \times \widehat{M}$.

Based on the defining concept for the quantum theta function (40), one can define the quantum theta function Θ_D in the noncommutative $\mathbb{T}^4$ case as

$$_D\langle f, f\rangle = \frac{1}{\sqrt{2^2\det(\operatorname{Im} T)}}\Theta_D, \tag{43}$$

for f given by (41) and T given by Ω that appeared in (23). According to (36), the $\mathcal{S}(\mathcal{D})$-valued inner product (43) can be written as

$$_D\langle f, f\rangle = \sum_{h\in D}\langle f, \pi_h f\rangle e_{D,\alpha}(h). \tag{44}$$

In [5], Manin showed that the quantum theta function defined in (43) is given by

$$\Theta_D = \sum_{h\in D}\mathrm{e}^{-\frac{\pi}{2}H(\underline{h},\underline{h})}e_{D,\alpha}(h), \tag{45}$$

where

$$H(\underline{g}, \underline{h}) \equiv \underline{g}^t(\operatorname{Im} T)^{-1}\underline{h}^*$$

with $\underline{h}^* = \overline{T}h_1 + h_2$ denoting the complex conjugate of $\underline{h}$. At the same time, it also satisfies a quantum version of the translation action for the classical theta functions [3]:

$$^\forall g \in D, \qquad C_g e_{D,\alpha}(g) x_g^*(\Theta_D) = \Theta_D, \tag{46}$$

where C_g is defined by

$$C_g = \mathrm{e}^{-\frac{\pi}{2}H(\underline{g},\underline{g})}$$

and the action of x_g^*, 'quantum translation', is given by

$$x_g^*(e_{D,\alpha}(h)) = \mathrm{e}^{-\pi H(\underline{g},\underline{h})} e_{D,\alpha}(h). \tag{47}$$

In [3], Manin has also required that the factor C_g, $g \in D$ appearing in the quantum translation x_g^* has to satisfy the following relation under a combination of quantum translations for consistency:

$$\frac{C_{g+h}}{C_g C_h} = \mathcal{T}_g(h)\alpha(g,h). \tag{48}$$

Here $\alpha(g,h)$ is the cocycle appearing in (35) and $\mathcal{T}_g(h)$ is a generalized expression of the factor that appears by quantum translation,

$$x_g^*(e_{D,\alpha}(h)) \equiv \mathcal{T}_g(h) e_{D,\alpha}(h). \tag{49}$$

The proof of the functional relation (46) in this embedding case with quantum translation (47) is shown in [5], in which the complex structure T is given by Ω in (23).

3.2.2. Embedding into lattice. We now turn to the second embedding case of nonzero q, where we do not have holomorphic vectors, the so-called theta vectors, once we assign a complex structure over the whole $\mathbb{T}^4$. A way-out from this difficulty turned out to be introducing a complex structure partially, i.e., only over the continuous subspace of the embedding space. As a result of this we got the function $f(s, n_1, n_2)$ (33) as an element of the module relevant to the nonzero q embedding.

With the function $f(s, n_1, n_2)$, we now evaluate the quantum theta function, and see whether it satisfies the functional relation for 'quantum translation'. We first define the quantum theta function *à la* (43) for $\mathbb{T}^4$ in the $q = 2$ case:

$$\frac{1}{\sqrt{2\,\mathrm{Im}\,T}}\hat{\Theta}_D = {}_D\langle f, f\rangle, \tag{50}$$

where T is a 'complex structure' over the continuous part of the embedding space to be specified below. We then show that the above-defined quantum theta function satisfies a functional relation *à la* (46) with modified quantum translation,

$$\forall g \in D, \qquad \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*(\hat{\Theta}_D) = \hat{\Theta}_D, \tag{51}$$

where $\hat{C}_g$, $\hat{x}_g^*$ are to be defined below.

To evaluate the quantum theta function (50), we calculate the scalar product inside the summation in (44) first. For that we first write the action of the operator π_h on f:

$$\pi_h f(s, n_1, n_2) = \mathrm{e}^{2\pi\mathrm{i}(w_{h2}s+t_1n_1+t_2n_2)+\pi\mathrm{i}(w_{h1}w_{h2}+m_1t_1+m_2t_2)} f(s + w_{h1}, n_1 + m_1, n_2 + m_2), \tag{52}$$

where $h \in D$ is given by

$$h = (w_{h1}, w_{h2}, m_1, m_2, t_1, t_2) \in \mathbb{R} \times \mathbb{R}^* \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{T} \times \mathbb{T}.$$

Then,

$$\langle f, \pi_h f\rangle = \sum_{n_1,n_2\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{\pi[\mathrm{i}\frac{\tau}{\theta_1}s^2 - \frac{1}{\theta_2}(n_1^2+n_2^2)]}\, \mathrm{e}^{-2\pi\mathrm{i}(w_{h2}s+t_1n_1+t_2n_2)-\pi\mathrm{i}(w_{h1}w_{h2}+m_1t_1+m_2t_2)}$$
$$\times\, \mathrm{e}^{\pi[-\mathrm{i}\frac{\bar{\tau}}{\theta_1}(s+w_{h1})^2 - \frac{1}{\theta_2}[(n_1+m_1)^2+(n_2+m_2)^2]]}$$

$$= \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi[\frac{\mathrm{Im}\,\tau}{\theta_1}s^2+\mathrm{i}\frac{\bar{\tau}}{\theta_1}w_{h1}s+\mathrm{i}w_{h2}s]-\mathrm{i}\pi[\frac{\bar{\tau}}{\theta_1}(w_{h1})^2+w_{h1}w_{h2}]}$$

$$\times\, \mathrm{e}^{-\frac{\pi}{\theta_2}(m_1^2+m_2^2)-\pi\mathrm{i}(m_1t_1+m_2t_2)} \sum_{n_1,n_2\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi}{\theta_2}(n_1^2+n_2^2)+2\pi\mathrm{i}[n_1(-t_1+\frac{\mathrm{i}m_1}{\theta_2})+n_2(-t_2+\frac{im_2}{\theta_2})]}$$

$$= b_{t_1,m_1}b_{t_2,m_2}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi[\frac{\mathrm{Im}\,\tau}{\theta_1}s^2+\mathrm{i}\frac{\bar{\tau}}{\theta_1}w_{h1}s+\mathrm{i}w_{h2}s]-\mathrm{i}\pi[\frac{\bar{\tau}}{\theta_1}(w_{h1})^2+w_{h1}w_{h2}]}, \tag{53}$$

where

$$b_{t_j,m_j} = \mathrm{e}^{-\frac{\pi}{\theta_2}m_j^2-\pi\mathrm{i}m_jt_j}\theta\left(\tau = \frac{2i}{\theta_2}, z = -t_j + \frac{im_j}{\theta_2}\right), \qquad j = 1, 2. \tag{54}$$

Here, $\theta(\tau, z)$ is the classical theta function defined by

$$\theta(\tau, z) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau n^2+2\pi\mathrm{i}nz}, \qquad \text{for} \quad \tau, z \in \mathbb{C}.$$

In order to facilitate the integration part, we denote the integrand as

$$\mathrm{e}^{-\pi[q(s)+l_{w_h}(s)+\widetilde{C}_{w_h}]}$$

with

$$q(s) = 2(\mathrm{Im}\,T)s^2, \qquad l_{w_h}(s) = 2\mathrm{i}(T^*w_{h1} + w_{h2})s, \qquad \widetilde{C}_{w_h} = iw_{h1}(T^*w_{h1} + w_{h2}),$$

where

$$T = \frac{\tau}{\theta_1}.$$

Using the relation

$$q(s + \lambda_{w_h}) - q(\lambda_{w_h}) = q(s) + l_{w_h}(s)$$

with

$$\lambda_{w_h} \equiv \frac{\mathrm{i}}{2}(\mathrm{Im}\,T)^{-1}\underline{w_h}^*,$$

the integration becomes

$$\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-\pi(q(s)+l_{w_h}(s)+\widetilde{C}_{w_h})} = \mathrm{e}^{-\pi(\widetilde{C}_{w_h}-q(\lambda_{w_h}))}\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-\pi q(s+\lambda_{w_h})} = \frac{1}{\sqrt{2\,\mathrm{Im}\,T}}\mathrm{e}^{-\pi(\widetilde{C}_{w_h}-q(\lambda_{w_h}))}.$$

With a straightforward calculation one can check that

$$\widetilde{C}_{w_h} - q(\lambda_{w_h}) = \tfrac{1}{2}H(\underline{w_h}, \underline{w_h}).$$

Thus the quantum theta function $\hat{\Theta}_D$ is given by

$$\hat{\Theta}_D = \sum_{h\in D}\widetilde{b}_h\, \mathrm{e}^{-\frac{\pi}{2}H(\underline{w_h},\underline{w_h})}e_{D,\alpha}(h), \tag{55}$$

where

$$\widetilde{b}_h = \prod_{j=1}^{2} b_{t_j,m_j} \tag{56}$$

with b_{t_j,m_j} given in (54).

To be consistently maintaining the symmetry property of the classical theta function under lattice translation, the above given quantum theta function should satisfy the functional relation under 'quantum translation' (51),

$$^{\forall}g \in D, \qquad \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*(\hat{\Theta}_D) = \hat{\Theta}_D,$$

and the consistency condition (48) for $\hat{C}_g$. The above relation is satisfied if we assign

$$\hat{C}_g = \widetilde{b}_g \,\mathrm{e}^{-\frac{\pi}{2}H(\underline{w_g},\underline{w_g})}, \tag{57}$$

and $\hat{x}_g^*$ is defined by

$$\hat{x}_g^*(e_{D,\alpha}(h)) = \hat{\mathcal{T}}_g(h)e_{D,\alpha}(h) \tag{58}$$

with

$$\hat{\mathcal{T}}_g(h) = \frac{\hat{C}_{g+h}}{\hat{C}_g\hat{C}_h\alpha(g,h)}. \tag{59}$$

Here we note that the quantum translations are not additive in this case:

$$\hat{x}_{g_1}^* \cdot \hat{x}_{g_2}^*(e_{D,\alpha}(h)) \neq \hat{x}_{g_1+g_2}^*(e_{D,\alpha}(h)). \tag{60}$$

On the other hand, the quantum translations in the Manin's case (x_g^*), (47), are additive:

$$x_{g_1}^* \cdot x_{g_2}^*(e_{D,\alpha}(h)) = x_{g_1+g_2}^*(e_{D,\alpha}(h)). \tag{61}$$

Now, it is easy to show the relation (51):

$$\begin{aligned}
\hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*(\hat{\Theta}_D) &= \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*\left(\sum_{h\in D}\widetilde{b}_h \,\mathrm{e}^{-\frac{\pi}{2}H(\underline{w_h},\underline{w_h})}e_{D,\alpha}(h)\right)\\
&= \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*\left(\sum_{h\in D}\hat{C}_h e_{D,\alpha}(h)\right)\\
&= \sum_{h\in D}\hat{C}_g\hat{C}_h e_{D,\alpha}(g)\hat{\mathcal{T}}_g(h)e_{D,\alpha}(h)\\
&= \sum_{h\in D}\hat{C}_{g+h}e_{D,\alpha}(g+h) = \hat{\Theta}_D,
\end{aligned}$$

where we used the relation (57) in the second step and the relation (59) together with the cocycle condition (35) in the last step.

4. Conclusion

In this paper, we study the theta vector and the corresponding quantum theta function in the embedding into the lattice for the noncommutative 4-torus.

While the theta vector exists in the embedding into the vector space case ($\mathbb{R}^p$ type), it does not exist in the embedding into the lattice case ($\mathbb{Z}^q$ type). And thus holomorphic theta vectors only exist for the vector space part in the case of mixed embedding ($\mathbb{R}^p \times \mathbb{Z}^q$ type). In general, the modules from embeddings including the lattice part are not fully holomorphic. Manin constructed the quantum theta functions only with holomorphic modules. Therefore, it is natural to ask whether one can construct the quantum theta function satisfying the Manin's requirement with the partially holomorphic modules in the mixed embedding case.

It turns out that these non-holomorphic modules also satisfy the requirement of the quantum theta function of Manin. We show this explicitly for the noncommutative 4-torus case with embedding into $\mathbb{R} \times \mathbb{Z}^2$. However, we note a feature that is different among the two quantum theta functions. In our quantum theta function constructed with a partially holomorphic module, two consecutive 'quantum translations' are not additive, while those in the Manin's are additive. This happens due to the consistency condition between quantum translation and the cocycle condition (48). The same holds for the quantum theta functions constructed with modules from embeddings into lattice ($\mathbb{Z}^q$) part only. This is due to the structure of the quantum theta function shown in (55) and that of the coefficient of the quantum translation, (57). Both of them consist of a direct product of the contributions from the two parts, one from the embedding into vector space and the other from the embedding into lattice.

In conclusion, we show explicitly that the quantum theta function that Manin defined can be constructed with any choice of the following embeddings, (1) into vector space times lattice, (2) into vector space, (3) into lattice, for the noncommutative 4-torus. We expect that this will hold for higher dimensional noncommutative tori.

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